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LINEAR PURSUIT PROBLEM UNDER LOCAL CONVEXITY CONDITIONS. SOLUTION OF THE SYNTHESIS EQUATION

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We derive conditions sufficient for the completion of pursuit by the time-independent feedback principle; the paper relates closely to the investigations in [1 – 7].

1. Let a linear pursuit problem in the n-dimensional Euclidean space R be described a) by the linear vector differential equation

$$dz / dt = Cz - u + v \tag{1.1}$$

where C is an nth-order constant square matrix; $u = u(t) \in P$ and $v = v(t) \in Q$ are vector-valued functions, measurable for $t \geqslant 0$, called the controls of the players (the pursuer and the pursued, respectively); $P \subset R$ and $Q \subset R$ are convex compacta;

b) by a terminal set M representable in the form $M = M_0 + W_0$, where M_0 is a linear subspace of space R, W_0 is some compact convex set in a subspace L which is the orthogonal complement to M_0 in R. By π we denote the operator of orthogonal projection onto L; we denote the dimension of L by ν and the unit sphere in L by K. We assume that $\nu \geqslant 2$. We denote the matrix e^{tC} by $\Phi(t)$. Every Carathéodory-solution $z(t)$ [1], $T_1 \leqslant t \leqslant T_2$, of Eq. (1.1) with the initial condition $z(T_1) = z_0$ is called a motion and denoted $z(t) = z(t;\ T_1, z_0, u, v, T_2)$.

The pursuer's aim is to bring point z onto set M; the pursued tries to prevent this. We say that the pursuit from a point z_0 can be concluded in a time $t(z_0)$ if there exists a vector-valued function $u(z) \in P$ (called the "synthesis"), defined on the whole space R, such that for arbitrary pursued's control $v(t)$, the pursuer by applying the control

$u(z(t))$ at every instant, can ensure that point z hits onto set M in a time not exceeding the number $t(z_0)$. We note that the solution of the pursuit problem with the aid of synthesis $\boldsymbol{u}(\boldsymbol{z}, \boldsymbol{t})$, depending on time, appears in a number of papers (for example, see [2, 3]).

2. We assume that Conditions 1 – 3 of [6, 7], whose notation we retain in the present paper, are fulfilled for problem (1.1). By Lemmas 1 qnd 2 we mean the corresponding lemmas from [7]. The following lemma was proved in [3].

Lemma 3. The function $T(z)$ is lower-semicontinuous on R.

It is easily verified that $(z, T(z)) \in D$ for any z such that $0 < T(z) < +\infty$.

Condition 4. Let $z \in R$ be such that $0 < T = T(z) < +\infty$ and

$$\frac{\partial\lambda(z,T)}{\partial t} = \frac{\partial^2\lambda(z,T)}{\partial t^2} = 0$$

then the function $\partial^2\lambda(z, t) / \partial t^2$ is differentiable at the point $(z, T(z))$, and

$$\partial^3\lambda(z, T) / \partial t^3 > 0 \tag{2.1}$$

$$\frac{\partial^3\lambda(z,T)}{\partial t^3} - \left(\frac{\partial^3\lambda(z,T)}{\partial z \partial t^2} \cdot G(z,T)\right) > 0 \tag{2.2}$$

Note. If in relations (2.1), (2.2) the sign of strict inequality is replaced by the sign of nonstrict inequality, these relations turn into the necessary conditions for the optimality [6, 7] of time $T(z)$.

Let $z \in R$, $0 < T(z) < +\infty$. We define a function $\varphi(z) \in K$ as follows: $\varphi(z) = \psi(z, T(z))$, i.e. (see [6, 7]), as $\varphi(z)$ we take the vector occurring in the relation

$$\pi\Phi(T(z))z = W(T(z), \varphi(z))$$

and we set $u(z) = u(T(z), \varphi(z))$. However, if $T(z) = 0$ or $T(z) = +\infty$, we set $u(z) = u^*$, where u^* is some fixed vector from P.

3. We now examine the vector differential equation (the synthesis equation)

$$dz / dt = Cz - u(z) + v_0(t) \tag{3.1}$$

where $v_0(t)$ is an arbitrary control of the pursued. Any absolutely continuous function $z(t)$ satisfying this equation for almost all values of t is called a solution of Eq. (3.1) (a Carathéodory solution) [1]).

Theorem. Suppose that Conditions 1 – 4 are fulfilled for problem (1.1). Let $z_0 \in R$, $0 < T_0 = T(z_0) < +\infty$. Then, for arbitrary pursued's control $v_0(t)$ the solution $z(t)$ of Eq. (3.1) with initial value $z(0) = z_0$ exists on some interval $[0, \delta]$, depending on control $v_0(t)$, such that $\delta \leqslant T(z_0)$ and $z(\delta) \in M$. Thus it is possible to complete the pursuit from point z_0 within time $T(z_0)$.

The theorem's proof is carried out separately for three cases.

1. Let $\partial\lambda(z_0, T_0) / \partial t > 0$. Then (see [7]) the functions $T(z)$ and $\varphi(z)$ are continuously differentiable in some neighborhood of point z_0 and, consequently (see Condition 1), we can find $\delta_1 > 0$ such that the solution $z(t)$ of Eq. (3.1) $(z(0) = z_0)$ exists and is unique for any control $v_0(t)$, $0 \leqslant t \leqslant \delta_1$, and

$$dT(z(t)) / dt \leqslant -1$$

almost everywhere on $[0, \delta_1]$. So that the inequality

$$0 < T(z(t_2)) \leqslant T(z(t_1)) - (t_2 - t_1) < +\infty \qquad (3.2)$$

is fulfilled for any t_1 and t_2 such that $0 \leqslant t_1 \leqslant t_2 \leqslant \delta_1$.

2. Let $\partial\lambda(z_0, T_0) / \partial t = 0$, $\partial^2\lambda(z_0, T_0) / \partial t^2 < 0$. Let us show that for arbitrary pursued's control $v_0(t)$ we can find $\delta_1 > 0$, depending on this control, such that a solution $z(t)$ of Eq. (3.1) $(z(0) = z_0)$, satisfying relation (3.2), exists (but not necessarily unique) on the interval $[0, \delta_1]$. We choose $\Delta > 0$ so small that the inequality $T(z(t)) \geqslant {}^1/_2 T_0$, $0 \leqslant t \leqslant \Delta$ is fulfilled for any motion $z(t; 0, z_0, u, v, \Delta)$. This is possible by virtue of Lemma 3. Here we can assume that

$$\Delta \leqslant \delta^* = \min \delta(t), \quad {}^1/_2 T_0 \leqslant t \leqslant T_0$$

where $\delta(t)$ is the continuous positive function given by Lemma 2 in [6]. Let m be an arbitrary positive integer. We determine the vector-valued function $z_m(t)$, $0 \leqslant t \leqslant \Delta$ by induction in the following way: $z_m(0) = z_0$, while on each of the intervals $[\beta_m{}^k, \beta_m^{k+1})$; $0 \leqslant k \leqslant 2^m - 1$, $\beta_m{}^k = 2^{-m} k\Delta$ we set

$$z_m(t) = z(t; \beta_m{}^k, z_m(\beta_m{}^k), u_m, v_0, \beta_m^{k+1})$$
$$u_m(t) \equiv u_m{}^k(t) = u(T(z_m(\beta_m{}^k)) - (t - \beta_m{}^k), \varphi(z_m(\beta_m{}^k)));$$
$$\beta_m{}^k \leqslant t < \beta_m^{k+1}$$

where v_0 is the control $v_0(t)$, $0 \leqslant t \leqslant \Delta$.

Since $\Delta \leqslant \delta^*$, in accordance with the alternative in [6]

$$T(z_m(t)) \leqslant T(z_m(\beta_m{}^k)) - (t - \beta_m{}^k), \quad t \in [\beta_m{}^k, \beta_m^{k+1}]. \qquad (3.3)$$

The set of motions is compact [2]. Therefore, a subsequence $z_i^*(t) = z_{m_i}(t)$ and a control $u_0(t)$, $0 \leqslant t \leqslant \Delta$ exist such that

$$z_i^*(t) \rightrightarrows z(t; 0, z_0, u_0, v_0, \Delta) = z(t)$$

on the interval $[0, \Delta]$ (here $\rightrightarrows$ is the symbol for uniform convergence). Here [1]

$$\lim_{i\to\infty} z_i^*(t_i) = z(t)$$

for any $t \in [0, \Delta]$ and for any sequence $\{t_i\}_{i=1}^{\infty} \subset [0, \Delta]$, converging to t. By N we denote the set of all sequences of binary rational numbers of the interval [0, 1]. We can prove that there exists $\delta_1 \in (0, \Delta]$ such that

$$\lim_{i\to\infty} T(z_i^*(\alpha_i\Delta)) = T(z(\tau)) \qquad (3.5)$$

holds for any $\tau \in [0, \delta_1]$ and for any sequence $\{\alpha_i\}$ from N converging to τ / Δ.

By $\theta(\tau)$ we denote

$$\theta(\tau) = \sup \overline{\lim_{i\to\infty}} T(z_i^*(\alpha_i\Delta))$$

(sup in the right hand side is taken over all sequences $\{\alpha_i\}$ from N converging to τ/Δ); then equality (3.5) is equivalent to the equality (see Lemma 3)

$$\theta(\tau) = T(z(\tau))$$

for all $\tau \in [0, \delta_1]$. We shall prove this. From (3.3) it follows that the inequality

$$T(z_i^*(\alpha\Delta)) \leqslant T_0 - \alpha\Delta$$

is valid for all i for any binary rational $\alpha \in [0, 1]$, whence

$$\theta(\tau) \leqslant T_0 - \tau \tag{3.6}$$

Further, since $\lambda(z_i^*(\alpha_i\Delta), t) < 0$, $0 \leqslant t < T(z_i^*(\alpha_i\Delta))$ and $\lambda(z_i^*(\alpha_i\Delta), T(z_i^*(\alpha_i\Delta))) = 0$, we have (see (3.4))

$$\lambda(z(\tau), t) \leqslant 0, \qquad 0 \leqslant t \leqslant \overline{\lim_{i\to\infty}}\, T(z_i^*(\alpha_i\Delta))$$

$$\lambda(z(\tau), \overline{\lim_{i\to\infty}}\, T(z_i^*(\alpha_i\Delta))) = 0$$

and, consequently,

$$\lambda(z(\tau), t) \leqslant 0, \quad 0 \leqslant t \leqslant \theta(\tau), \quad \lambda(z(\tau), \theta(\tau)) = 0 \tag{3.7}$$

Therefore, if $T(z(\tau)) < \theta(\tau)$, then necessarily

$$\frac{\partial}{\partial t}\lambda(z(\tau), T(z(\tau))) = 0 \tag{3.8}$$

If now we assume that there exists a sequence

$$\{\tau_i\}_{i=1}^{\infty} \subset [0, \Delta], \ \lim_{i\to\infty} \tau_i = 0 \ \text{ such that } \ T_i = T(z(\tau_i)) < \theta(\tau_i) = \theta_i,$$

we obtain

$$T_0 \leqslant \lim T_i \leqslant \overline{\lim}\, T_i \leqslant \overline{\lim}\, \theta_i \leqslant T_0$$

by using inequality (3.6) and the lower semicontinuity of $T(z)$.

In this connection (Lemma 2), for all sufficiently large i the continuous second derivative of function $\lambda(z(\tau_i), t)$ exists on the interval $[T_i, \theta_i]$. By virtue of (3.7), (3.8) we then can find $\xi_i \in [T_i, \theta_i]$ such that

$$\partial^2\lambda(z(\tau_i), \xi_i) / \partial t^2 = 0 \tag{3.9}$$

We have

$$T_0 \leqslant \underline{\lim}\, T_i \leqslant \underline{\lim}\, \xi_i \leqslant \overline{\lim}\, \xi_i \leqslant \lim \theta_i \leqslant T_0$$

Hence, by virtue of the continuity of the second derivatives of the function $\lambda(z, t)$ at the point (z_0, T_0) and of equality (3.9), we obtain

$$0 = \lim_{i\to\infty} \partial^2\lambda(z(\tau_i), \xi_i) / \partial t^2 = \partial^2\lambda(z_0, T_0) / \partial t^2$$

A contradiction! Equality (3.5) is proved.

Since the relation

$$\varphi(z_i^*(\alpha_i\Delta)) = \psi(z_i^*(\alpha_i\Delta), T(z_i^*(\alpha_i\Delta)))$$

is fulfilled for any sequence $\{\alpha_i\}$ from N converging to $\tau / \Delta \in [0, \delta / \Delta]$, we obtain

$$\varphi(z(\tau)) = \psi(z(\tau), T(z(\tau))) = \lim_{i\to\infty} \varphi(z_i^*(\alpha_i\Delta)) \tag{3.10}$$

by using Lemma 1 and relations (3.4), (3.5). We can show that

$$\lim_{i\to\infty} u_{m_i}(\tau) = u(T(z(\tau)), \varphi(z(\tau))) = u(z(\tau)) \tag{3.11}$$

everywhere on $[0, \delta_1]$.

In fact, let $\tau \in [0, \delta_1]$ and let the sequence $\alpha_i = \beta_{m_i}^{k_i}$ be such that

$$\beta_{m_i}^{k_i} \leqslant \tau < \beta_{m_i}^{k_i+1}$$

Then by the definition of function u_m

$$u_{m_i}(\tau) = u(T(z_i^*(\alpha_i\Delta)) - (\tau - \alpha_i\Delta), \varphi(z_i^*(\alpha_i\Delta)))$$

Since $\alpha_i\Delta \to \tau$, we obtain (3.11) from formulas (3.5), (3.10) and from Condition 1 (the continuity of $u(r, \varphi)$).

By the Lebesgue theorem we have, therefore,

$$z(t)=\lim_{i\to\infty} z_i^*(t)=z_0+\lim_{i\to\infty}\int_0^t \{Cz_i^*(\tau)-u_{m_i}(\tau)+v_0(\tau)\}\,d\tau=$$

$$z_0+\int_0^t \{Cz(\tau)-u(z(\tau))+v_0(\tau)\}\,d\tau$$

for any $t\in[0,\ \delta_1]$ and, consequently,

$$dz(t)\,/\,dt=Cz(t)-u(z(t))+v_0(t)$$

almost everywhere on $[0,\ \delta_1]$, i.e. $z(t)$ is a solution of Eq. (3.1) and $z(0)=z_0$. From inequality (3.3) it follows directly that for all sufficiently large i the inequality

$$T(z_i^*(\alpha^{(2)}\Delta))\leqslant T(z_i^*(\alpha^{(1)}\Delta))-(\alpha^{(2)}\Delta-\alpha^{(1)}\Delta)$$

holds for any binary rational $\alpha^{(1)},\ \alpha^{(2)},\ 0\leqslant\alpha^{(1)}\leqslant\alpha^{(2)}\leqslant 1$. Choosing the sequences $\{\alpha_i^{(1)}\}$ and $\{\alpha_i^{(2)}\}$ from N converging, respectively, to $t_1\,/\,\Delta$ and $t_2\,/\,\Delta$, $0\leqslant t_1\leqslant t_2\leqslant\delta_1$, we obtain relation (3.2) by using (3.5).

3. Let $\partial\lambda(z_0,\ T_0)\,/\,\partial t=\partial^2\lambda(z_0,\ T_0)\,/\,\partial t^2=0$. We can prove that for any control $v_0(t)$ of the pursued we can find $\delta_1>0$ such that the solution $z(t)$ of Eq.(3.1) $(z(0)=z_0)$, satisfying relation (3.2), exists in the interval $[0,\ \delta_1]$.

It can be verified that in this case all the reasonings for the second case can be repeated verbatim up to the assumption on the nonfulfillment of equality (3.5) and the resulting assumption of equality (3.9). Furthermore, as soon as relation (3.5) is proved we can repeat verbatim all the reasonings which followed it in the proof, including the obtaining of inequality (3.2). Thus, it remains only to show that equality (3.9) leads to a contradiction.

Since $\xi_i\to T_0$ as $i\to\infty$, in accordance with Condition 4

$$0=\frac{\partial^2\lambda(z(\tau_i),\ \xi_i)}{\partial t^2}=A(\xi_i-T_0)+(B[z(\tau_i)-z_0])+\beta_i \qquad (3.12)$$

where A is the number $\partial^3\lambda(z_0,\ T_0)\,/\,\partial t^3$, B is the vector $\partial^3\lambda(z_0,\ T_0)\,/\,\partial z\partial t^2$ and $\beta_i\,((\xi_i-T_0)^2+|z(\tau_i)-z_0|^2)^{-1/2}\to 0$ as $i\to\infty$. We divide (3.12) by (ξ_i-T_0) and we pass to the limit with respect to i. We obtain

$$0=A+\lim_{i\to\infty}\left(B\ \frac{z(\tau_i)-z_0}{\tau_i}\right)\frac{\tau_i}{\xi_i-T_0}+\lim\ \frac{\beta_i}{\xi_i-T_0} \qquad (3.13)$$

Noting that (see (3.6))

$$0<\frac{\tau_i}{T_0-\xi_i}\leqslant 1 \qquad (3.14)$$

and that

$$\overline{\lim_{i\to\infty}}\ \frac{|z(\tau_i)-z_0|}{\tau_i}\leqslant|Cz_0|+\max_{u\in p,v\in Q}(|u|+|v|) \qquad (3.15)$$

we have

$$\frac{\beta_i}{\xi_i-T_0}=-\frac{\beta_i}{((\xi_i-T_0)^2+|z(\tau_i)-z_0|^2)^{1/2}}\left\{1+\left|\frac{z(\tau_i)-z_0}{\tau_i}\right|^2\left(\frac{\tau_i}{T_0-\xi_i}\right)^2\right\}^{1/2}\to 0$$

It remains to find the limit of the second term in (3.13). The sequence $z_i^*(t)$ converges to $z(t)$ uniformly on $[0,\ \Delta]$; therefore, from it we can pick out a subsequence (we retain the same notation for it because we could have taken precisely this subsequence as z_i^* (t) right from the start) such that

$$|z(t) - z_i^*(t)| \leqslant \tau_i^2, \quad 0 \leqslant t \leqslant \Delta$$

Then, obviously,

$$\lim_{i\to\infty} \frac{z(\tau_i) - z_0}{\tau_i} = \lim_{i\to\infty} \frac{z_i^*(\tau_i) - z_0}{\tau_i} = \lim_{i\to\infty} \frac{1}{\tau_i} \int_0^{\tau_i} \{Cz_i^*(t) - u_{m_i}(t) + v_0(t)\}\, dt$$

By virtue of (3.4),

$$\frac{1}{\tau_i} \int_0^{\tau_i} Cz_i^*(t)\, dt \to Cz_0$$

Further, since (see (3.3)) $T(z_i^*(t)) \leqslant T_0 - t,\ t \in [0, \tau_i]$,

$$\lim_{i\to\infty} \max_{t\in[0,\tau_i]} |T(z_i^*(t)) - T_0| = 0$$

on the basis of the lower semicontinuity of function $T(z)$ and, consequently (see (3.10), (3.11)),

$$\lim_{i\to\infty} \max_{t\in[0,\tau_i]} |u_m(t) - u(z_0)| = 0$$

Since

$$\frac{1}{\tau_i} \int_0^{\tau_i} v_0(t)\, dt = v_i \in Q$$

(set Q is a convex compactum), without loss of generality (choosing, if necessary, a subsequance from $\{\tau_i\}$ and denoting it once again by $\{\tau_i\}$) we can assume that $v_i \to v^* \in Q$ as $i \to \infty$. So that

$$\frac{z(\tau_i) - z_0}{\tau_i} \to Cz_0 - u(z_0) + v^* \qquad (3.16)$$

If $v^* = v(T_0, \varphi(z_0))$, then, making use of (3.14) and passing, if necessary, once more to a subsequence, we can assume that

$$\frac{\tau_i}{T_0 - \xi_i} \to \alpha^* \in [0, 1], \quad i \to \infty$$

and, consequently (see (3.13)),

$$0 = A - \alpha^*(B\cdot G(z_0 T(z_0))) = (1 - \alpha^*) A + \alpha^*(A - (B\cdot G(z_0 T_0))) > 0$$

A contradiction.

However, if $v^* \neq v(T(z_0), \varphi(z_0))$, then, using the fact that $\theta_i \to T_0$, we obtain (in accordance with the assumption that $\partial\lambda(z_0, T_0) / \partial t = 0$)

$$0 = \lambda(z(\tau_i), \theta_i) = (F \cdot [z(\tau_i) - z_0]) + \varepsilon_i \qquad (3.17)$$
$$\varepsilon_i((\theta_i - T_0)^2 + |z(\tau_i) - z_0|^2)^{-1/2} \to 0, \quad i \to \infty$$

where F is the vector $\partial\lambda(z_0, T_0) / \partial z$. Dividing (3.17) by $T_0 - \theta_i$ and noting that $0 < \tau_i / T_0 - \theta_i \leqslant 1$ (see (3.6)), we obtain

$$0 = (F\cdot\{Cz_0 - u(z_0) + v^*\})\, \alpha_*, \quad \alpha_* = \lim_{i\to\infty} \tau_i / T_0 - \theta_i$$

by using (3.15) and passing, if necessary, once again to a subsequence. Since

$$-\left(F \cdot \{Cz_0 - u(z_0) + v^*\} = \frac{\partial\lambda(z_0, T_0)}{\partial t} + (\varphi(z_0)\cdot\Phi(T_0)\{v^* - v(T_0, \varphi(z_0))\}\right) < 0$$

by virtue of Lemma 2 and of Condition 1, we have $\alpha_* = 0$. Hence follows $\alpha^* = 0$ in accordance with the inclusion $\xi_i \in [T_i, \theta_i]$; consequently, the limit of the second term in (3.13) equals zero. Hence, $A = 0$. This contradicts (2.1). Equality (3.5) is proved.

Thus, as soon as the theorem's hypotheses are fulfilled, for any control $v_0(t)$, $0 \leqslant t \leqslant T_0$, we can find $\delta_1 > 0$ such that the solution $z(t)$ of Eq. (3.1) $(z(0) = z_0)$, satisfying condition (3.2), exists on the interval $[0, \delta_1]$. By virtue of the lower semicontinuity of function $T(z)$ we can, by choosing $\delta_1 > 0$ sufficiently small, assume that

$$T(z(t)) > 0, \qquad t \in [0, \delta_1] \tag{3.18}$$

We fix $v_0(t)$, $0 \leqslant t < T_0$. Let Z be the set of all solutions $z(t)$, $0 \leqslant t \leqslant \delta_1$, $z(0) = z_0$ of Eq. (3.1), each defined on its own interval $[0, \delta_1]$ and satisfying inequalities (3.2) and (3.18); on set Z we define an order relation by setting $z'(t) \prec z''(t)$ if and only if the interval $[0, \delta_1']$ on which solution $z'(t)$ is defined is contained in the interval $[0, \delta_1'']$ on which solution $z''(t)$ is defined and, in addition $z'(t) \equiv z''(t)$, $0 \leqslant t \leqslant \delta_1$. It is easy to verify that every linearly ordered subset of Z has a majorant, so that a maximal element $z_0(t)$, $0 \leqslant t \leqslant \delta_0$, exists by Zorn's lemma (see [8]).

Let us show that $z_0(\delta_0) \in M$. In fact, since

$$0 \leqslant T(z_0(\delta_0)) \leqslant T_0 - \delta_0 < +\infty$$

by virtue of (3.2), all the reasonings of cases 1 – 3 are applicable to the point $z_0' = z_0(\delta_0)$ if $T(z_0(\delta)) > 0$, and, consequently, the solution $z'(t)$, defined on some interval $[\delta_0, \delta_0 + \varepsilon]$, $\varepsilon > 0$, of Eq. (3.1) with initial condition $z'(\delta_0) = z_0'$ exists, satisfying inequality (3.2) for any $\delta_0 \leqslant t_1 \leqslant t_2 \leqslant \delta_0 + \varepsilon$ and the inequality $T(z'(t)) > 0$, $t \in [\delta_0, \delta_0 + \varepsilon]$. If now

$$z(t) = \begin{cases} z_0(t), & 0 \leqslant t \leqslant \delta_0 \\ z'(t), & \delta_0 \leqslant t \leqslant \delta_0 + \varepsilon \end{cases}$$

then, obviously, $z_0(t) \prec z(t)$, which contradicts the maximality of $z_0(t)$. The theorem is completely proved.

We note that the solution of the synthesis equation in the general case is not unique. In [9] [sic!] there exists a condition (*) under which uniqueness takes place, and the results of the theorem just proved are applicable to the pursuit problem.

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ON THE REDUCTION OF DIFFERENTIAL EQUATIONS TO THE NORMAL FORM BY AN ANALYTIC TRANSFORMATION

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We treat the problem of reducing a system of nth-order ordinary differential equations to normal form in a neighborhood of a singular point in the presence or absence of resonances. We have shown that such a reduction is possible in the class of analytic transformations if the original system admits of an analytic symmetry group of specified dimension.

We examine the real nth-order autonomous system

$$x^{\cdot} = f(x), \qquad f(0) = 0 \tag{1}$$

We assume that the vector-valued function $f(x)$ is analytic in a neighborhood of the point $x = 0$, that among the eigenvalues λ_k of the linear part there are no multiple ones, and that only a finite number m of linearly independent formal operators

$$X = \sum \xi_i(x)\,\partial/\partial x_i \not\equiv \mu L$$

exist ($\xi_i(x)$ are formal power series), commuting with the shift operator

$$L = \Sigma f_i(x)\partial / \partial x_i$$

along the trajectories of system (1), $[L, X] = 0$. With system (1) we associate a finite-dimensional maximal group G of analytic transformations of a neighborhood of point $x = 0$ preserving this system, namely, an analytic symmetry group (cf. [1]) (the elements of algebra $\mathbf{L}$ of group G are infinitesimal analytic operators). Let l be the number of independent resonance relations $\lambda_1 k_1 + \ldots + \lambda_n k_n = 0$ $(0 \leqslant l \leqslant n-1,\ k_i \geqslant 0)$.

Theorem. For system (1) to be reducible to normal form in a neighborhood of point $x = 0$ by an analytic transformaltion, it is sufficient, and for $l = 0, 1$ also necessary, to fulfil the condition $\dim G = m$.

Proof. Let φ be an invertible transformation reducing system (1) to the normal form

$$y_i^{\cdot} = y_i p_i, \qquad i \leqslant n \tag{2}$$

It is easy to verify that system (2) admits of a group with the operators

$$Y = \sum_{i=1}^{n} \alpha_i y_i \frac{\partial}{\partial y_i} \tag{3}$$